ALL PROOFS OF TRIGONOMETRY

XI-MATHEMATICS

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PROOFS OF TRIGONOMETRY

Subject Specialist:
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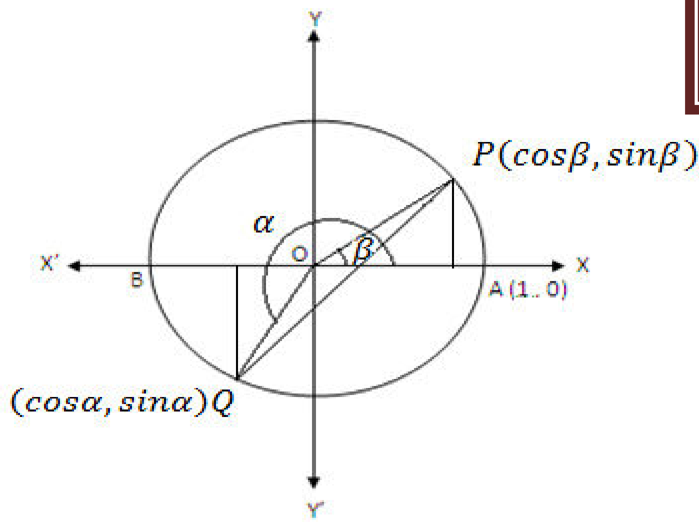
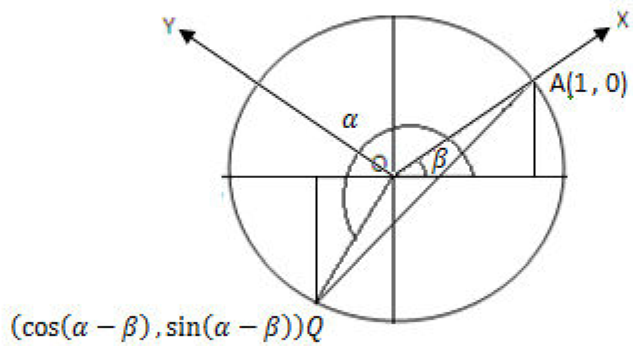
"Merging man and Math"

Fundamental Law:

Consider a unit circle with centre at O(0,0) as shown in figure.



Let P(cosβ, sinβ) and Q(cosα, sinα) be any two points in unit circle.
We have distance formula as:
d = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Here $\overline{PQ}$ = $\sqrt{(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2}$ ------------------(1)
Now rotate the axes so that the positive direction of X-axis passes through the point P.



Then with respect to this coordinate system, the coordinates of P and Q become (1,0) and (cos$(\alpha-\beta)$, sin$(\alpha-\beta)$) respectively.
So, $\overline{PQ}$ = $\sqrt{[\cos(\alpha-\beta) - 1]^2 + [(\sin(\alpha-\beta) - 0)]^2}$ ------------------(2)
Comparing (1) and (2), we get
$\sqrt{(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2}$ = $\sqrt{[\cos(\alpha-\beta) - 1]^2 + [(\sin(\alpha-\beta) - 0)]^2}$
or $(cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2 = [\cos(\alpha-\beta) - 1]^2 + [(\sin(\alpha-\beta) - 0)]^2$
or $\cos^2\alpha - 2\cos\alpha.\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha.\sin\beta + \sin^2\beta = \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)$
or $\sin^2\alpha + \cos^2\alpha - 2\cos\alpha.\cos\beta - 2\sin\alpha.\sin\beta + \sin^2\beta + \cos^2\beta = \sin^2(\alpha-\beta) + \cos^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)$
or $1 - 2\cos\alpha.\cos\beta - 2\sin\alpha.\sin\beta + 1 = 1 + 1 - 2\cos(\alpha-\beta)$
or $-2\cos\alpha.\cos\beta - 2\sin\alpha.\sin\beta = -2\cos(\alpha-\beta)$
Dividing by -2, we get
or $\cos\alpha.\cos\beta + \sin\alpha.\sin\beta = \cos(\alpha-\beta)$

Hence, $\boxed{\cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta}$

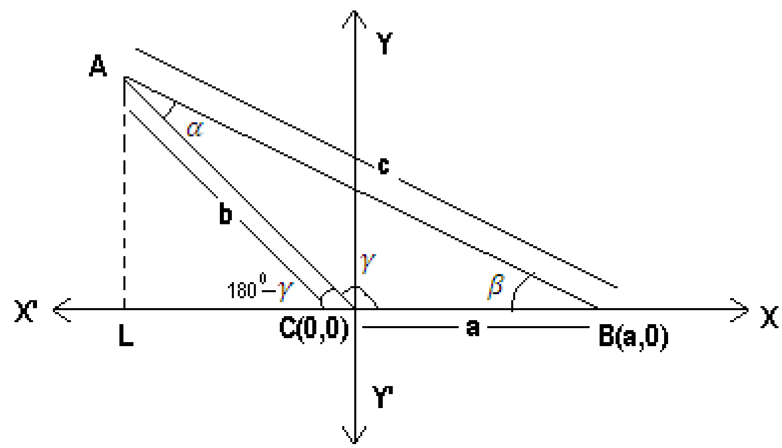
Law of sin:

Statement:

Law of Sine states that in a triangle measure of the sides are proportional to sine of the angle opposite to theses sides as: $\frac{a}{sin\alpha} = \frac{b}{sin\beta} = \frac{c}{sin\gamma}$

Proof:

We place a ΔABC in x-, y- coordinate system such that C(0,0) is at the origin and B(a,0) on positive x-axis as shown in the figure.



As $\cos(180^0 - \gamma) = \frac{base}{per} = \frac{\overline{CL}}{\overline{AC}}$

$\text{Cos}180^0.\cos\gamma - \sin180^0.\sin\gamma = \frac{\overline{CL}}{b}$

$-\cos\gamma = \frac{\overline{CL}}{b}$

$\boxed{\overline{CL} = -b.\sin\gamma}$

$\sin(180^0 - \gamma) = \frac{per}{hyp} \frac{\overline{AL}}{\overline{AC}}$

$\sin180^0.\cos\gamma - \cos180^0.\sin\gamma = \frac{\overline{AL}}{b}$

$\sin\gamma = \frac{\overline{AL}}{b}$

$\boxed{\overline{AL} = b.\sin\gamma}$

So the coordinates of A are (b.cosγ, b.sinγ)

Now, If B is taken at the origin

So, similarly coordinates of A are (c.cosβ, c.sinβ)

Hence y-coordinate $\overline{AL}$, in both cases is same.

$\overline{AL} = \overline{AL}$

b.sinγ = c.sinβ

$\frac{b}{sin\beta} = \frac{c}{sin\gamma}$ ------------(1)

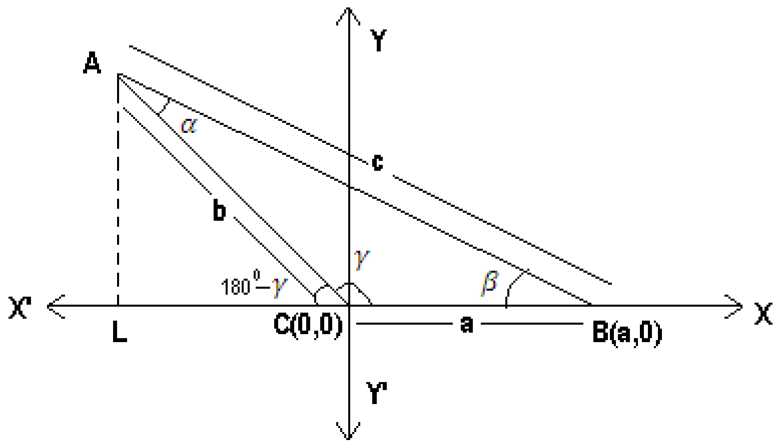
$\frac{b}{sin\beta} = \frac{a}{sin\alpha}$ ------------(2)

Comparing (1) and (2), we get

$$\boxed{\frac{a}{sin\alpha} = \frac{b}{sin\beta} = \frac{c}{sin\gamma}}$$

Law of Cosine: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof: We place a ΔABC in x-, y- coordinate system such that C(0,0) is at the origin and B(a,0) on positive x-axis as shown in the figure.



We have, $\cos(180^0 - \gamma) = \frac{base}{hyp} = \frac{\overline{CL}}{\overline{AC}}$

$\cos180^0.\cos\gamma + \sin180^0.\sin\gamma = \frac{\overline{CL}}{b}$

$-\cos\gamma = \frac{\overline{CL}}{b}$

$\boxed{\overline{CL} = -b.\cos\gamma}$

And $\sin(180^0 - \gamma) = \frac{per}{hyp} = \frac{\overline{AL}}{\overline{AC}}$

$\sin180^0.\cos\gamma - \cos180^0.\sin\gamma = \frac{\overline{AL}}{b}$

$\sin\gamma = \frac{\overline{AL}}{b}$

$\boxed{\overline{AL} = b.\sin\gamma}$

So, coordinates of A are (b.cosγ, b.sinγ)

We have distance formula as:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $\overline{AB} = c = \sqrt{(b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2}$

$c = \sqrt{b^2\cos^2\gamma - 2ab\cos\gamma + a^2 + b^2\sin^2\gamma}$

$c = \sqrt{b^2\cos^2\gamma + b^2\sin^2\gamma - 2ab\cos\gamma + a^2}$

$c = \sqrt{b^2(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma + a^2}$

$c = \sqrt{b^2(1) - 2ab\cos\gamma + a^2}$

Squaring both sides

$c^2 = b^2 - 2ab\cos\gamma + a^2$

And hence, $\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma}$

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Prove that: $\Delta = \frac{1}{2} b^2 \frac{\sin\alpha.\sin\gamma}{\sin\alpha}$

Proof: We have law of sin as: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

So $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow a = \frac{b.\sin\alpha}{\sin\beta}$

And $\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} \Rightarrow c = \frac{b.\sin\gamma}{\sin\beta}$

Area of $\Delta = \frac{1}{2}ac.\sin\beta$

$= \frac{1}{2} \times \frac{b.\sin\alpha}{\sin\beta} \times \frac{b.\sin\gamma}{\sin\beta} \times \sin\beta$

Hence, $\boxed{\Delta = \frac{1}{2} b^2 \frac{\sin\alpha.\sin\gamma}{\sin\alpha}}$

Prove that: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof: We have $\Delta = \frac{1}{2}ac.\sin\beta$

$\Delta = \frac{1}{2} ac.\sin2(\frac{\beta}{2})$

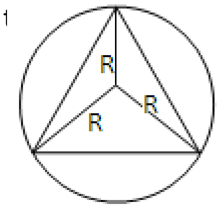
$\Delta = \frac{1}{2} ac \times 2\sin\frac{\beta}{2}.\cos\frac{\beta}{2}$ [using: sin2x = 2sinxcosx]

$= ac\sqrt{\frac{(s-a)(s-c)}{ac} \times \frac{s(s-b)}{ac}}$

$= \frac{ac}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

$$\boxed{\Delta = \sqrt{s(s-a)(s-b)(s-c)}}$$

Circum-circle: Circle circumscribing Δ is called the circum-circle OR the circle touching the vertices of Δ is called circum-circle. Its centre is known as circum centre and radius is known as circum radius denoted by R.



Prove that: R = $\frac{a}{2sin\alpha} = \frac{b}{2sin\beta} = \frac{c}{2sin\gamma}$ and hence show that: R = $\frac{abc}{4\Delta}$

Proof:

Let 'O' be the circum-centre of the circle of ΔABC. Join O and B and then produce $\overline{BO}$ to D. Join C and D. Then $\overline{BD}$ = 2R.

If the Δ is acute then m<A = α = m<D.

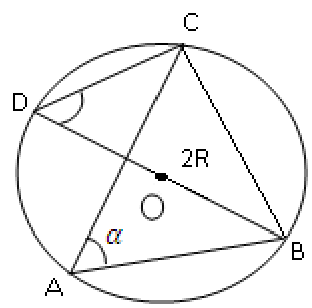
In right ΔBCD

$\sin\alpha = \frac{perp}{hyp} = \frac{\overline{BC}}{\overline{BD}} = \frac{a}{2R}$

$\sin\alpha = \frac{a}{2R}$

$2R\sin\alpha = a$

Hence, $\boxed{R = \frac{a}{2sin\alpha}}$



Now, $R = \frac{a}{2sin\alpha}$

$R = \frac{a}{2 \times 2sin\frac{\alpha}{2}.cos\frac{\beta}{2}}$

$R = \frac{a}{4 \times \sqrt{\frac{(s-b)(s-c)\times s(s-a)}{bc \times bc}}}$

$R = \frac{a}{4 \times \frac{\Delta}{bc}}$

$R = \frac{a \times bc}{4 \times \Delta}$

$\boxed{R = \frac{abc}{4\Delta}}$

In-circle: Circle touching the sides of a Δ is called in-circle. It's centre is known as in-centre and radius is known as in-radius denoted by 'r'.

Proof: Consider a triangle ΔABC.

'I' is the centre of the Δ. Then $\overline{IE} = \overline{IF} = \overline{ID}$ = r

Join $\overline{AC}$, $\overline{IB}$, $\overline{IA}$.

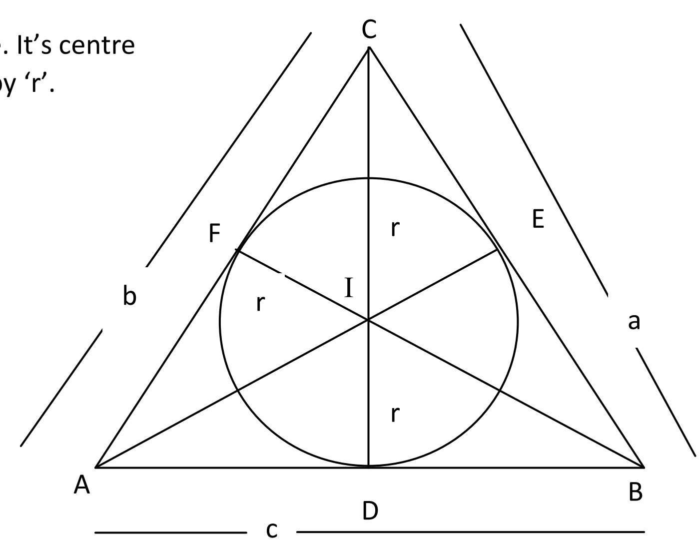
Here ΔABC = ΔABI + ΔBCI + ΔCAI

$\Delta = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br$

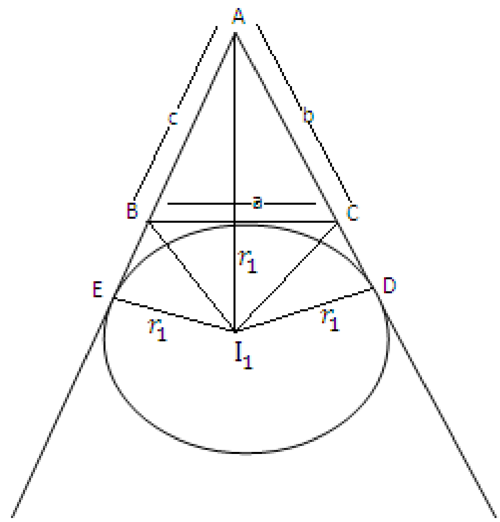
$\Delta = \frac{1}{2}r(a + b + c)$

$\Delta = rs$ [using: $s = \frac{a+b+C}{2}$]

Hence, $\boxed{r = \frac{\Delta}{s}}$



Prove that: $r_1 = \frac{\Delta}{s-a}$



Proof: Consider a ΔABC in which the e-circle is opposite to vertex A whose radius is r_1 and centre of circle is I_1. Join I_1A, I_1B, I_1C.

$$\text{Then, } \Delta ABC = \Delta I_1CA + \Delta I_1BA - \Delta I_1BC$$
$$= \frac{1}{2}br_1 + \frac{1}{2}cr_1 - \frac{1}{2}ar_1$$
$$= \frac{1}{2}r_1(b+c-a) \text{ -----------(1)}$$

$$\text{As } s = \frac{a+b+c}{2}$$
$$2s = a+b+c$$
$$2s - 2a = a+b+c-2a$$
$$2(s-a) = b+c-a$$

$$(1) \Rightarrow \quad \Delta = \frac{1}{2}r_1[2(s-a)]$$
$$\Delta = \frac{1}{2}r_1[2(s-a)]$$
$$\Delta = r_1(s-a)$$
$$\Delta = r_1(s-a)$$
$$\text{Hence, } r_1 = \frac{\Delta}{s-a}$$

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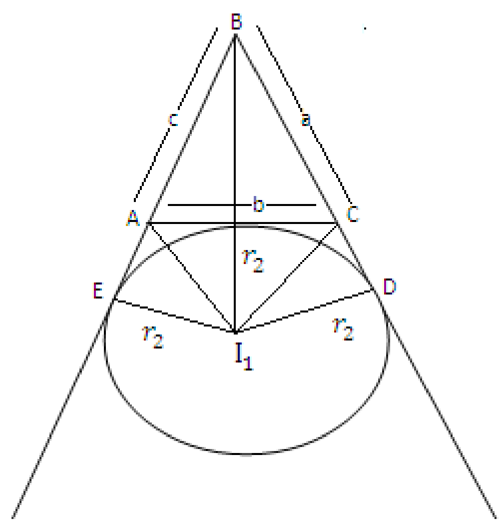
Prove that: $r_2 = \frac{\Delta}{s-b}$



Proof: Consider a ΔABC in which the e-circle is opposite to vertex B whose radius is r_2 and centre of circle is I_2. Join I_2A, I_2B, I_2C.

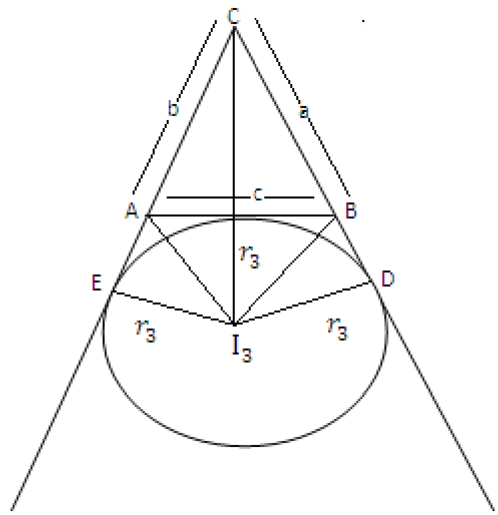
$$\text{Then, } \Delta ABC = \Delta I_2AB + \Delta I_2BC - \Delta I_2AC$$
$$= \frac{1}{2}cr_2 + \frac{1}{2}ar_2 - \frac{1}{2}br_2$$
$$= \frac{1}{2}r_2(c+a-b) \text{ -----------(1)}$$

$$\text{As } s = \frac{a+b+c}{2}$$
$$2s = a+b+c$$
$$2s - 2b = a+b+c-2b$$
$$2(s-b) = a+c-b$$

$$(1) \Rightarrow \quad \Delta = \frac{1}{2}r_2[2(s-b)]$$
$$\Delta = \frac{1}{2}r_2[2(s-b)]$$
$$\Delta = r_2(s-b)$$
$$\Delta = r_2(s-b)$$
$$r_2 = \frac{\Delta}{s-b}$$

Prove that: $r_3 = \frac{\Delta}{s-c}$

Proof: Consider a ΔABC in which the e-circle is opposite to vertex C whose radius is r_3 and centre of circle is I_3. Join I_3A, I_3B, I_3C.



Then, $\Delta ABC = \Delta I_3CA + \Delta I_3BC - \Delta I_3AB$

$= \frac{1}{2}br_3 + \frac{1}{2}ar_3 - \frac{1}{2}cr_3$

$= \frac{1}{2}r_3(b + a - c)$ -----------(1)

As $s = \frac{a+b+c}{2}$

$2s = a + b + c$

$2s - 2c = a + b + c - 2c$

$2(s - c) = a + b - c$

(1) $\Rightarrow$ $\Delta = \frac{1}{2}r_3[2(s - c)]$

$\Delta = \frac{1}{2}r_3[2(s - c)]$

$\Delta = r_3(s - c)$

$\Delta = r_3(s - c)$

$r_3 = \frac{\Delta}{s-c}$

Prove that: $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Proof:

As Cosine law says, $a^2 = b^2 + c^2 - 2bc.\cos\alpha$

So, $2bc.\cos\alpha = b^2 + c^2 - a^2$

$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$

Subtracting from both sides

$1 - \cos\alpha = 1 - \frac{b^2+c^2-a^2}{2bc}$

$2\sin^2\frac{\alpha}{2} = \frac{2bc-b^2-c^2+a^2}{2bc}$ [using: $1 - \cos x = 2\sin^2\frac{x}{2}$]

$2\sin^2\frac{\alpha}{2} = \frac{a^2-b^2+2bc-c^2}{2bc}$

$2\sin^2\frac{\alpha}{2} = \frac{a^2-(b^2-2bc+c^2)}{2bc}$

$2\sin^2\frac{\alpha}{2} = \frac{a^2-(b-c)^2}{2bc}$

$2\sin^2\frac{\alpha}{2} = \frac{(a+b-c)[a-(b-c)]}{2bc}$ ----------(1)

$2\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a+c-b)}{2bc}$

As $s = \frac{a+b+c}{2}$

$2s = a + b + c$

$2s - 2c = a + b + c - 2c$

$\boxed{2(s - c) = a + b - c}$

And $2s - 2b = a + b + c - 2b$

$\boxed{2(s - b) = a + c - b}$

(1) $\Rightarrow$ $2\sin^2\frac{\alpha}{2} = \frac{2(s-b)\times 2(s-c)}{2bc}$

$\sin^2\frac{\alpha}{2} = \frac{(s-b)\times(s-c)}{bc}$

Hence, $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Prove that: $\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{bc}}$

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Proof:

As Cosine law says, $b^2 = a^2 + c^2 - 2ac \times \cos\beta$

So, $2ac \times \cos\beta = a^2 + c^2 - b^2$

$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$

Subtracting from both sides

$1 - \cos\beta = 1 - \frac{a^2+c^2-b^2}{2ac}$

$2\sin^2\frac{\beta}{2} = \frac{2ac-a^2-c^2+b^2}{2ac}$ [using: $1 - \cos x = 2\sin^2\frac{x}{2}$]

$2\sin^2\frac{\beta}{2} = \frac{b^2-a^2+2ac-c^2}{2bc}$

$2\sin^2\frac{\beta}{2} = \frac{b^2-(a^2-2ac+c^2)}{2ac}$

$2\sin^2\frac{\beta}{2} = \frac{b^2-(a-c)^2}{2bc}$

$2\sin^2\frac{\beta}{2} = \frac{(b+a-c)[b-(a-c)]}{2ac}$ ----------(1)

$2\sin^2\frac{\beta}{2} = \frac{(a+b-c)(b+c-a)}{2ac}$

As $s = \frac{a+b+c}{2}$

$2s = a + b + c$

$2s - 2c = a + b + c - 2c$

$2(s - c) = a + b - c$

And $2s - 2a = a + b + c - 2a$

$2(s - a) = b + c - a$

(1) $\Rightarrow$ $2\sin^2\frac{\beta}{2} = \frac{2(s-c) \times 2(s-a)}{2ac}$

$\sin^2\frac{\beta}{2} = \frac{(s-a) \times (s-c)}{ac}$

Hence, $\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$

Prove that: $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Proof:

As Cosine law says, $c^2 = a^2 + b^2 - 2ab \times \cos\gamma$

So, $2ac \times \cos\gamma = a^2 + b^2 - c^2$

$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$

Subtracting from both sides

$1 - \cos\gamma = 1 - \frac{a^2+b^2-c^2}{2ab}$

$2\sin^2\frac{\gamma}{2} = \frac{2ab-a^2-b^2+c^2}{2ab}$ [using: $1 - \cos x = 2\sin^2\frac{x}{2}$]

$2\sin^2\frac{\gamma}{2} = \frac{c^2-a^2+2ab-b^2}{2ab}$

$2\sin^2\frac{\gamma}{2} = \frac{c^2-(a^2-2ab+b^2)}{2ab}$

$2\sin^2\frac{\gamma}{2} = \frac{c^2-(a-b)^2}{2ab}$

$2\sin^2\frac{\gamma}{2} = \frac{(c+a-b)[c-(a-b)]}{2ab}$ ----------(1)

$2\sin^2\frac{\gamma}{2} = \frac{(c+a-b)(c+b-a)}{2ab}$

As $s = \frac{a+b+c}{2}$

2s = a + b + c

2s – 2b = a + b + c – 2b

2(s – b) = a + c -b

And 2s – 2a = a + b + c – 2a

2 (s - a) = b + c – a

(1) ⇒ $2\sin^2\frac{\gamma}{2} = \frac{2(s-b)\times 2(s-a)}{2ab}$

$\sin^2\frac{\gamma}{2} = \frac{(s-a)\times(s-b)}{ab}$

Hence, $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Prove that: $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Proof:

As Cosine law says, $a^2 = b^2 + c^2 - 2bc \times \cos\alpha$

So, $2bc \times \cos\alpha = b^2 + c^2 - a^2$

$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$

Adding 1 both sides

$1 + \cos\alpha = 1 + \frac{b^2+c^2-a^2}{2bc}$

$2\cos^2\frac{\alpha}{2} = \frac{2ab+b^2+c^2-a^2}{2bc}$ [using: $1 + \cos x = 2\cos^2\frac{x}{2}$]

$2\cos^2\frac{\alpha}{2} = \frac{(b+c)^2-a^2}{2bc}$

$2\cos^2\frac{\alpha}{2} = \frac{(b+c+a)(b+c-a)}{2bc}$ -----------------(1)

As $s = \frac{a+b+c}{2}$

2s = a + b + c

And 2s – 2a = a + b + c – 2a

2 (s - a) = b + c – a

(1) ⇒ $2\cos^2\frac{\alpha}{2} = \frac{2s\times 2(s-a)}{2bc}$

$\cos^2\frac{\alpha}{2} = \frac{s\times(s-b)}{bc}$

Hence, $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Prove that: $\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$

Proof:

As Cosine law says, $b^2 = a^2 + c^2 - 2ac \times \cos\beta$

So, $2ac \times \cos\beta = a^2 + c^2 - b^2$

$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$

Adding 1 both sides

$1 + \cos\beta = 1 + \frac{a^2 + c^2 - b^2}{2ac}$

$2\cos^2\frac{\beta}{2} = \frac{2ac + a^2 + c^2 - b^2}{2ac}$ [using: $1 + \cos x = 2\cos^2\frac{x}{2}$]

$2\cos^2\frac{\beta}{2} = \frac{(a+c)^2 - b^2}{2ac}$

$2\cos^2\frac{\beta}{2} = \frac{(a+c+b)(a+c-b)}{2ac}$ ------------------(1)

As $s = \frac{a+b+c}{2}$

$\boxed{2s = a + b + c}$

And $2s - 2b = a + b + c - 2b$

$\boxed{2(s - b) = b + c - b}$

(1) ⇒ $2\cos^2\frac{\beta}{2} = \frac{2s \times 2(s-b)}{2ac}$

$\cos^2\frac{\beta}{2} = \frac{s \times (s-b)}{ac}$

Hence, $\boxed{\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}}$

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Prove that: $\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Proof:

As Cosine law says, $c^2 = a^2 + b^2 - 2ab \times \cos\gamma$

So, $2ab \times \cos\gamma = a^2 + b^2 - c^2$

$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$

Adding 1 both sides

$1 + \cos\gamma = 1 + \frac{a^2 + b^2 - c^2}{2ab}$

$2\cos^2\frac{\gamma}{2} = \frac{2ab + a^2 + b^2 - c^2}{2ab}$ [using: $1 + \cos x = 2\cos^2\frac{x}{2}$]

$2\cos^2\frac{\gamma}{2} = \frac{(a+b)^2 - c^2}{2ab}$

$2\cos^2\frac{\gamma}{2} = \frac{(a+b+c)(a+b-c)}{2ab}$ ------------------(1)

As $s = \frac{a+b+c}{2}$

$\boxed{2s = a + b + c}$

And $2s - 2c = a + b + c - 2c$

$\boxed{2(s - c) = a + b - c}$

(1) ⇒ $2\cos^2\frac{\gamma}{2} = \frac{2s \times 2(s-c)}{2ab}$

$\cos^2\frac{\gamma}{2} = \frac{s \times (s-c)}{ab}$

Hence, $\boxed{\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}}$